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Damage mechanics approach to nonlinear behavior of FRP laminates with cracking layers

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Abstract—A damage mechanics model to predict the nonlinear behavior of laminated composites due to crack evolution is developed. We propose a new concept that a cracking layer can be replaced with an equivalent uniform *work-softening* layer. With this new concept, the constitutive equations for a cracking layer are constructed according to modern plasticity theory. A lamina damage surface, i.e. a threshold of crack evolution, is defined in the stress space of each lamina, and the constitutive equations for a cracking layer are constructed by applying the defined damage surface to the associated flow rule. Then, the derived constitutive equations for a cracking layer are introduced with classical lamination theory to predict laminate constitutive behavior in the damage process. Comparisons are made between the predictions and experimental results for some laminate configurations, and reasonable agreements are shown for all laminates.

Keywords: Damage mechanics; composite material; constitutive equation; matrix cracking; work-softening.

1. INTRODUCTION

Composite materials play an increasingly important role in industry today, mainly because of their high specific stiffness, strength and excellent fatigue resistance. However, since fiber-reinforced composites are being used increasingly in precision components, the problem of damage initiation and evolution in composites is becoming a more important subject.

Because of the inherent weakness of laminae in their transverse direction, unidirectional laminae are primarily utilized in the form of laminates consisting of differently oriented layers. Many experimental observations have confirmed that, in multi-angleply laminates, the initiation of damage is often by matrix cracks in the off-axis plies running parallel to the fiber directions in those plies. Also, formation

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of cracks causes deterioration of macroscopic material properties; therefore, the response of a laminated composite with cracking layers would be nonlinear after the appearance of cracks.

The damage development and the damage effects on the mechanical behavior of FRP laminates have been studied by many researchers, and various kinds of models have been developed. Among existing models, a damage mechanics approach, in which the effects of cracks are expressed in constitutive equations through a set of internal state variables, could be the most reasonable one when considering the macroscopic nonlinear behavior of a laminate due to crack evolution. This approach was first utilized for laminated composites by Talreja [1], followed by Chow and Wang [2], Allen *et al.* [3, 4], Joshi and Frantziskonis [5], and others. These models are successful in the sense that they can precisely predict current elastic response if the current values of internal state variables are known. However, in these models, the damage evolution law, the dependence of internal state variables on the loading history, was not fully developed.

In this work, the analytical model to predict the nonlinear stress–strain behavior of laminated composites due to matrix crack evolution is developed in the framework of continuum damage mechanics. The formulation is performed in two steps. In the first step, we focus exclusively on the cracking layer. A new concept that a cracked layer can be replaced with an equivalent uniform work-softening layer is proposed, and the analytical technique in plasticity theory is introduced to describe the constitutive behavior of a cracking layer. Appropriate expressions for the constitutive equations are obtained according to the associated flow rule, and the damage evolution law is reduced to the determination of a hardening modulus. It is shown that the variational analysis presented by Hashin [6] is available to obtain the appropriate form of the hardening modulus. In the second step, the derived constitutive equations for a cracking layer are incorporated into the conventional lamination theory to describe the nonlinear constitutive behavior of a laminated composite with cracking layers. Finally, in order to verify the validity of the present model, comparisons are made for $[0/\theta_n/0]$ type laminates between the predictions and experiments.

2. BASIC ASSUMPTION

Under service loading, subcritical damage in the form of matrix cracking, interlaminar delamination, fiber breaking, etc., is detected until the final catastrophic failure of its structural integrity. Many observations have confirmed that in multi-angleply laminates, matrix cracks running parallel to the fiber direction is the most frequently observed form of damage at the early stage of their service life. For this reason, hereafter, the formulation is restricted to matrix cracking damage, and the effects of other modes of damage are neglected.

As is confirmed by Reifsnider [7], matrix cracks tend to form periodic saturation patterns which he called the Characteristic Damage State (CDS). It is well

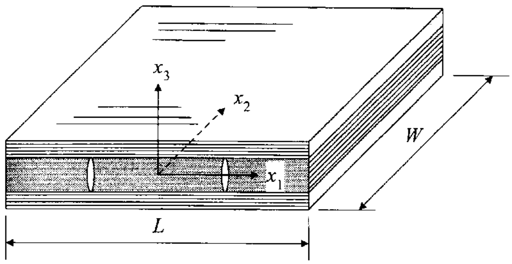


Figure 1. Coordinate system in a cracked laminate.

known that these crack formations cause redistribution of stresses and deterioration of macroscopic material properties. Among existing theoretical models to describe these phenomena, Nairn's model, which is an extension of Hashin's variational analysis, may be the most effective one. The stress field of a crossply laminate between two adjacent cracks under uniaxial tension is derived by Nairn as follows [8].

$$\begin{cases} \sigma_{11}^{(1)} = \sigma_T^{(1)}(1-\phi(x_1)), \\ \sigma_{13}^{(1)} = \sigma_T^{(1)} x_3 \dfrac{d\phi(x_1)}{dx_1}, \\ \sigma_{33}^{(1)} = \dfrac{1}{2}\sigma_T^{(1)}(ht^{(1)} - x_3^2)\dfrac{d^2\phi(x_1)}{dx_1^2}, \end{cases} \qquad \begin{cases} \sigma_{11}^{(2)} = \sigma_T^{(2)} + \dfrac{t^{(1)}}{t^{(2)}}\sigma_T^{(1)}\phi(x_1), \\ \sigma_{13}^{(2)} = \dfrac{t^{(1)}}{t^{(2)}}\sigma_T^{(1)}(h - x_3)\dfrac{d\phi(x_1)}{dx_1}, \\ \sigma_{33}^{(2)} = \dfrac{t^{(1)}}{2t^{(2)}}\sigma_T^{(1)}(h - x_3)^2\dfrac{d^2\phi(x_1)}{dx_1^2}, \end{cases} \quad (1)$$

where, and from now on, superscripts (1) and (2) refer to the 90° ply and the 0° ply, respectively, $\sigma_T^{(1)}$ and $\sigma_T^{(2)}$ denote the normal stresses in both layers if they are assumed to be undamaged, $t^{(1)}$ and $t^{(2)}$ ($h = t^{(1)} + t^{(2)}$) are the half of the thicknesses of respective plies and $\phi(x_1)$ represents the stress perturbation caused by the adjacent two cracks. The coordinate system is: x_1 is the fiber direction of the 0° ply, x_3 is the thickness direction, and the origin is placed at the center of two adjacent cracks (see Fig. 1).

With this approximate stress field, the effective compliance of each layer is defined in terms of the crack configuration. The strain energy stored in each layer at a given crack configuration (N crack spaces) under a given applied load are readily evaluated by introducing equation (1) to the appropriate energy integral expression for each ply [9].

$$U^{(1)} = \frac{1}{2}(P + P_t)^2 S_0 \Omega_1, \quad (2)$$

$$U^{(2)} = \frac{1}{2}(P + P_t)^2 S_0 \Omega_2 - P_t(P + P_t)S + \frac{P_t^2 L}{4t^{(2)} W E_L}, \quad (3)$$

where

$$\begin{cases} \Omega_1 = \dfrac{t^{(1)}E_T}{hE_C} - 2\dfrac{t^{(2)}E_L}{hE_C}K\sum_{i=1}^{N}\chi(\rho_i) + \dfrac{1}{2}K\sum_{i=1}^{N}\chi^{(1)}(\rho_i), \\ \Omega_2 = \dfrac{t^{(2)}E_L}{hE_C} + 2\dfrac{t^{(2)}E_L}{hE_C}K\sum_{i=1}^{N}\chi(\rho_i) + \dfrac{1}{2}K\sum_{i=1}^{N}\chi^{(2)}(\rho_i), \\ P_t = -2t^{(2)}WE_L\Delta\alpha\Delta T, \end{cases} \tag{4}$$

where summations cover N crack spaces (i refers to i-th crack space), P denotes the applied load, S_0 and S represent the compliance of the uncracked and cracked laminate, respectively, $\chi(\rho_i)$, $\chi^{(1)}(\rho_i)$ and $\chi^{(2)}(\rho_i)$ are defined [9] as functions of the nondimensional crack density ρ_i (90° ply thickness divided by the interval between two adjacent cracks), E_L and E_T represent the original lamina Young's modulus in longitudinal and transverse direction respectively, E_C is the initial undamaged laminate Young's modulus in the fiber direction of the outer plies, W and L are the laminate width and length, respectively, $\Delta\alpha$ is the difference between longitudinal and transverse thermal expansion coefficients, $\Delta T = T_S - T_0$ (T_S: room temperature, T_0: reference temperature at which residual strains are zero), and K is a constant determined by the original lamina elastic moduli and the thickness of each ply, etc.

By use of the effective compliance, the strain energy in each layer at residual stress free temperature can be written in another way as

$$U^{(1)}\big|_{\Delta T=0} = \frac{1}{2}P^{(1)^2}S^{(1)}, \tag{5}$$

$$U^{(2)}\big|_{\Delta T=0} = \frac{1}{2}P^{(2)^2}S^{(2)}, \tag{6}$$

where $S^{(1)}$ and $S^{(2)}$ denote the effective compliance of respective layer, and $P^{(1)}$ and $P^{(2)}$ are the average load carried by respective layer.

Considering the fact that the compliance is independent of temperature in the linear thermoelasticity theory, the expression of the effective compliance is obtained by equating equation (2) with (5) and equation (3) with (6) after setting $P_t = 0$ in equations (2) and (3). The resulting equations are given by

$$S^{(1)} = \frac{S^2}{S_0\Omega_1}, \tag{7}$$

$$S^{(2)} = \frac{S^2}{S_0\Omega_2}. \tag{8}$$

In the derivation of equations (7) and (8), the equivalence of strains ($P^{(1)}S^{(1)} = P^{(2)}S^{(2)} = PS$) is taken into account. Since S (the compliance of the cracked laminate) is derived from the knowledge of the distribution of crack spacings, the effective compliance of each layer is readily obtained in terms of the crack distribution using equations (7) and (8).

With the effective compliance just defined, the average stress–strain curve of a cracking layer is readily obtained through a simple tension test on a crossply laminate. The dependence of crack configurations on the applied load is first measured, then the applied load corresponding to each crack configuration is divided between each ply along with equations (7) and (8) to calculate the average stress of the cracking layer. Figure 2 shows the average stress–strain curve of the 90° layer in the lamina coordinate (x'_1: longitudinal direction, x'_2: transverse direction; here and from now on, primes are referred to the lamina principal coordinate). The material properties are summarized in Table 1. It is obvious that the average stress in the cracking layer (90° ply) continuously decreases after crack initiation, and it shows similar behavior to what we have called a work-softening material.

Although Fig. 2 is the result for crossply laminates, it would be plausible to assume that a cracked layer can be treated as a work-softening material in general laminate configurations.

Table 1.
Material properties of glass/epoxy Scotchply lamina

E_L (GPa)	E_T (GPa)	G_L (GPa)	G_T (GPa)	ν_L	ν_T
40.3	10.5	3. 78	4.58	0.326	0.380

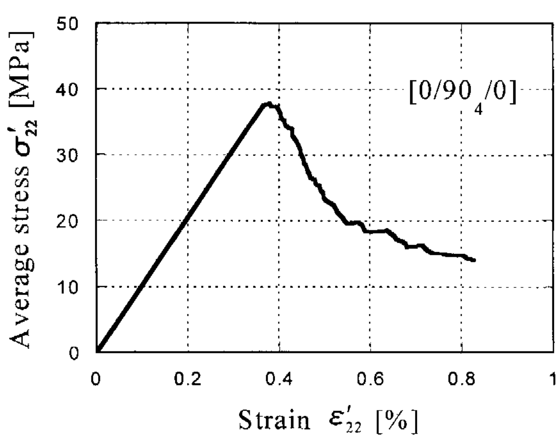


Figure 2. Average stress–strain curve of the 90° layer in GFRP crossply laminates fabricated by Scotchply prepreg.

3. FORMULATION OF CRACKED LAMINA CONSTITUTIVE EQUATIONS

3.1. Definition of a lamina damage surface

In order to formulate the constitutive equations for a cracking layer along with the plasticity theory, a lamina damage surface, which plays a similar role to a yield function in the plasticity theory, must be defined. Under membrane load, a matrix crack in laminates is generally caused by three inplane stress components in the lamina coordinate; σ'_{11}, σ'_{22} and σ'_{12}. However, for convenience, let us assume that σ'_{11} does not have a significant effect on crack formation. With this assumption, the specification of a lamina damage surface is reduced to the determination of an unknown function of σ'_{22} and σ'_{12} in the stress space.

For derivation of the appropriate form of an initial damage surface, let us first consider a lamina fracture surface. Following the conventional technique, a lamina fracture surface was investigated with the strength data of unidirectionally reinforced laminates with the fiber direction inclined at a different angle from the loading direction. Six kinds of specimens of GFRP unidirectional laminate $[\theta_6]$ ($\theta = 90$, 75, 60, 45, 30, 15) were tested in tension at a crosshead speed of 0.2 [mm/min]. The specimen dimensions are 280 (length) × 14 (width) [mm]. Figure 3 depicts the stress states corresponding to each laminate fracture. Each plot is the averaged value of five specimens. It is obvious that two stress components clearly interact with each other, and the fracture surface is approximately reproduced by an ellipsoid.

In general, an initial damage surface of a lamina is not identical to the lamina fracture surface because of the constraint effects from adjacent plies. However, for simplicity, let us assume that an initial damage surface has a similar outline to that of the fracture surface. This assumption implies that the constraint effects from adjacent layers are reflected through the scale change of the initial damage surface. The exact specification of the initial lamina damage surface is the subject of further investigations. From the above discussions, the initial damage surface of a lamina

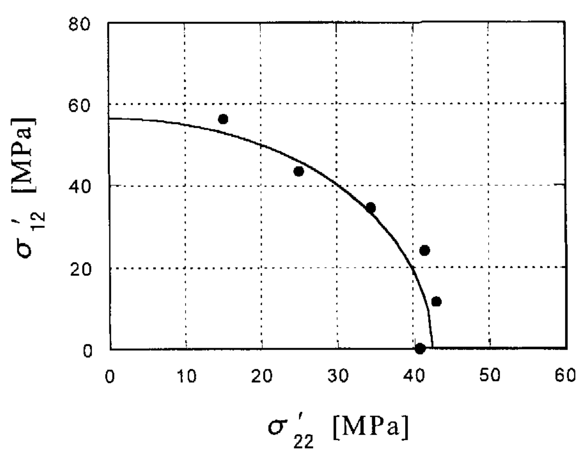


Figure 3. Fracture surface of glass/epoxy Scotchply lamina.

in the principal material coordinate is expressed by

$$f(\sigma'_{22}, \sigma'_{12}) = \left(\frac{\sigma'_{22}}{m_1}\right)^2 + \left(\frac{\sigma'_{12}}{m_2}\right)^2 - 1 = 0, \tag{9}$$

where m_1 and m_2 are material constants which depend on the laminate configuration. The constraint effects from adjacent layers are introduced to the theory through these constants.

For work-softening materials, a subsequent damage surface is not the same as the initial damage surface; for the special case that a damage surface is always equivalent to the initial damage surface, it is called perfect plasticity. According to the conventional plasticity theory, there are several models to describe a subsequent damage surface, for instance isotropic hardening or kinematic hardening, etc. The simplest case would be isotropic hardening. In the case of isotropic hardening, the subsequent damage surface in the lamina coordinate is given by

$$f(\sigma'_{22}, \sigma'_{12}) = \left(\frac{\sigma'_{22}}{m_1}\right)^2 + \left(\frac{\sigma'_{12}}{m_2}\right)^2 - g(\zeta) = 0, \tag{10}$$

where ζ is a measure of damage level in the cracked lamina. In this work, ζ has the same physical meaning as the number of cracks because we restrict ourselves to matrix cracking only. Moreover, $g(\zeta)$ is the scale function of the damage surface. For the case of a work-softening material, $g(\zeta)$ is a monotonic decreasing function of the damage level ζ.

In the following sections, the damage surface given by equation (10) is used to construct the constitutive equations of a cracking layer in general laminate configurations.

3.2. Formulation of constitutive equations for a cracking layer

In this section, with the assumption of the equivalence between a cracking layer and a work-softening material, the constitutive equations for a cracking layer are constructed according to the plasticity theory. In the plasticity theory, if a material has a flow potential which is equivalent to the yield function, the material is said to obey the associate flow rule, and the constitutive equations are expressed by [10]

$$d\sigma_{ij} = \left\{C_{ijkl} - \frac{1}{L}C_{ijmn}h_{mn}C_{pqkl}\frac{\partial f}{\partial \sigma_{pq}}\right\}d\varepsilon_{kl}, \tag{11}$$

where

$$L = H + C_{ijkl}h_{ij}h_{kl}, \quad H = (C_{ijkl}h_{ij}h_{kl})\frac{h_{mn}d\sigma_{mn}}{C_{pqrs}h_{pq}d\varepsilon^p_{rs}}, \quad h_{ij} = \frac{\partial f}{\partial \sigma_{ij}}, \tag{12}$$

where C_{ijkl} are the elastic moduli, ε^p_{ij} are the plastic strains, $f = 0$ is the yield surface, and H is called the hardening modulus. The substitution of equation (10) into equations (11) and (12) gives

$$d\sigma'_{ij} = \left\{ C_{ijkl} - \frac{1}{L} C_{ijmn} C_{pqkl} h_{mn} h_{pq} \right\} d\varepsilon'_{kl}, \tag{13}$$

where

$$L = H + C_{2222} h_{22}^2 + C_{1212} h_{12}^2, \tag{14}$$

$$H = \frac{\left(C_{2222} h_{22}^2 + C_{1212} h_{12}^2\right)\left(h_{22} d\sigma'_{22} + h_{12} d\sigma'_{12}\right)}{C_{2222} h_{22} d\varepsilon'^{i}_{22} + C_{1212} h_{12} d\varepsilon'^{i}_{12}}, \tag{15}$$

$$h_{11} = 0, \quad h_{22} = \frac{2\sigma'_{22}}{m_1^2}, \quad h_{12} = \frac{2\sigma'_{12}}{m_2^2}, \tag{16}$$

where inelastic strains (ε'^{i}_{ij}) are used in place of plastic strains (ε'^{p}_{ij}). It should be emphasized that inelastic strains in a cracking layer are defined in the loading stage only and are not equivalent to the true residual strains that appear when a cracked laminate is unloaded. As illustrated in Fig. 4, the real stress–strain curve of a cracking layer has sudden stress relaxation due to crack formation. The replacement of a cracking layer by a uniform layer corresponds to the replacement of the real stress–strain curve by the broken line in Fig. 4. Thus, inelastic strains are defined as the residual strains of the fictitious plastic material.

Among the above equations, the most dominant term is the hardening modulus H, which defines present damage evolution rate. Any other terms can be determined if a specific damage surface is selected. Therefore, the determination of H as a function of the damage level is needed to complete the constitutive equations for a cracking layer. In the following section, the experimental determination of the hardening modulus is discussed in detail.

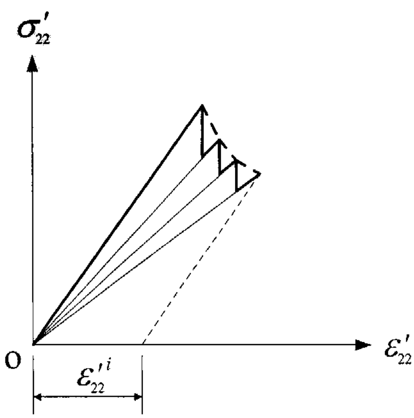


Figure 4. Replacement of a cracking layer with a plastic material.

3.3. Determination of a hardening modulus

In this section, the experimental procedures to obtain the appropriate expression of the hardening modulus are proposed with some fundamental assumptions. As shown in equation (15), the hardening modulus H for orthotropic laminae has a fairly complicated form, and it is impossible to determine H through simple experiments. However, according to the associated flow rule, H must satisfy the following relation.

$$d\varepsilon_{ij}^{\prime i} = \frac{1}{H} h_{kl} d\sigma_{kl}^{\prime} h_{ij}. \tag{17}$$

Equation (17) is simplified as follows when equation (10) is accepted as the damage surface.

$$d\vec{\varepsilon}^{\prime i} = \frac{1}{H}(\vec{h} \cdot d\vec{\sigma}^{\prime})\vec{h}, \tag{18}$$

where

$$d\vec{\varepsilon}^{\prime i} = (d\varepsilon_{22}^{\prime i}, d\varepsilon_{12}^{\prime i}), \quad d\vec{\sigma}^{\prime} = (d\sigma_{22}^{\prime}, d\sigma_{12}^{\prime}), \quad \vec{h} = (h_{22}, h_{12}). \tag{19}$$

Furthermore, taking the scalar product between $\vec{h}$ and both sides of equation (18), then solving the resulting equation about H, we can obtain the following expression.

$$H = h^2 \frac{d\sigma_T^{\prime}}{d\varepsilon^{\prime i}}, \tag{20}$$

where

$$d\varepsilon^{\prime i} = |d\vec{\varepsilon}^{\prime i}|, \quad d\sigma_T^{\prime} = \frac{|\vec{h} \cdot d\vec{\sigma}^{\prime}|}{|\vec{h}|}, \quad h = |\vec{h}|, \tag{21}$$

where $d\sigma_T^{\prime}$ is the projection of a stress increment vector to the normal direction of the present damage surface. It should be noted that $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$ does not stand for a differentiation, and it should be interpreted as the ratio of a stress increment to a strain increment. The reason is that $\sigma_T^{\prime}$ has no physical meaning, and there is no unique relationship between $\sigma_T^{\prime}$ and $\varepsilon^{\prime i}$. In equation (20), h is completely determined by the initial damage surface. On the contrary, $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$ is generally unknown.

To understand the implication of the term $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$, it is useful to consider the vicinity of the present damage surface. Figure 5 schematically depicts the stress and strain increments on the damage surface. In this figure, the incremental vector of inelastic strains is restricted to the normal direction of the present damage surface. This is a requirement from the normality rule in plasticity theory. As a result, $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$ is interpreted as the increment ratio in the normal direction of the present damage surface.

Generally, it is impossible to obtain $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$ for an arbitrary damage state; however, in the case of a crossply laminate under uniaxial load, $d\sigma_T^{\prime}/d\varepsilon^{\prime i}$ takes

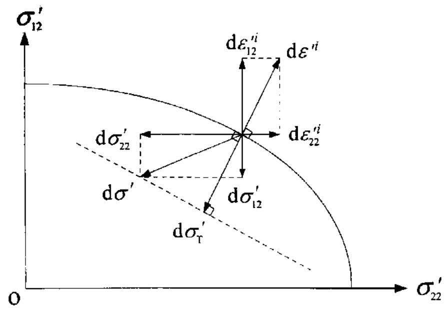


Figure 5. Stress and inelastic strain increments on a damage surface.

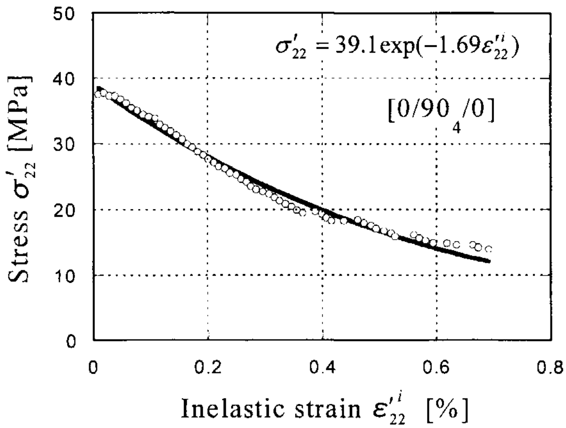


Figure 6. Relationship between stress and inelastic strain in a cracking layer of GFRP crossply laminates under uniaxial tension.

the following form.

$$\frac{d\sigma'_T}{d\varepsilon'^{i}} = \frac{d\sigma'_{22}}{d\varepsilon'^{i}_{22}}. \tag{22}$$

Equation (22) can be determined with the experiment on a crossply laminate. Using the stress–strain curve in Fig. 2, the relation between σ'_{22} and ε'^{i}_{22} is easily obtained. The derived relation between σ'_{22} and ε'^{i}_{22} is shown in Fig. 6. In this figure, the solid line is the approximation by the function shown in this figure. It should be noted that in this case, σ'_{22} and ε'^{i}_{22} are uniquely correlated. Therefore, we can obtain $d\sigma'_{22}/d\varepsilon'^{i}_{22}$ from the differentiation of the stress (σ'_{22}) with respect to the inelastic strain (ε'^{i}_{22}).

In more general cases such as multi-angleply laminates subjected to arbitrary membrane load, equation (22) no longer holds, and $d\sigma'_T/d\varepsilon'^{i}$ cannot be determined from experiments. However, if we adopt the assumption that $d\sigma'_T/d\varepsilon'^{i}$ is constant on the same damage surface, Fig. 6 can be used for general laminate configurations in an arbitrary loading path, so long as the same initial damage surface is used. That is, $d\sigma'_T/d\varepsilon'^{i}$ has a one-to-one relationship with the damage surface (the damage level), and it is always identical to $d\sigma'_{22}/d\varepsilon'^{i}_{22}$. Therefore, if the damage level is known, the

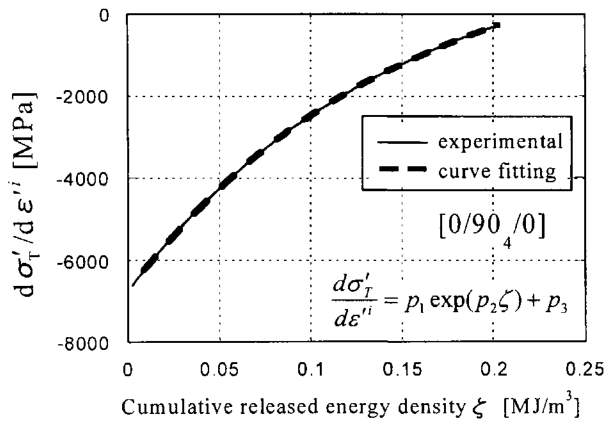


Figure 7. Damage evolution law of a cracking layer in GFRP laminates.

only thing to be done is to pick up the slope of the σ'_{22}-ε''^i_{22} curve at a corresponding damage level and use it as $d\sigma'_T/d\varepsilon''^i$. The problem is the definition of the internal state variable (ζ) which represents the present damage level. Although ε''^i_{22} is a good variable to represent the damage level of unidirectionally loaded crossply laminates, it is inadequate to take ε''^i_{22} as the measure of damage for more complicated cases. For a proper definition of ζ, we must start with a criterion for crack initiation.

We have already studied a crack initiation condition with crossply laminates, and experimentally confirmed that the energy release rate of a cracking layer works as the criterion for matrix crack onset [9]. Under this criterion, a crack will be formed if the released energy in a cracking layer due to new crack formation reaches a certain critical value. Consequently, the energy released by one crack formation remains constant, and the cumulative released energy has the same physical meaning as the number of cracks. From the above consideration, it should be concluded that the cumulative released energy in a cracking layer is the proper internal variable reflecting the present damage level.

Figure 6 is converted to Fig. 7 to show the dependence of $d\sigma'_T/d\varepsilon''^i$ $(= d\sigma'_{22}/d\varepsilon''^i_{22})$ on the cumulative released energy in a cracking layer. The cumulative released energy at a given number of cracks is computed by multiplying the total crack surface area and the critical energy release rate. The critical energy release rate used is 180 J/m^2. In this figure, for convenience, the cumulative released energy density in a cracking layer (the cumulative released energy per unit volume of a cracking layer: ζ) is used as the abscissa. The broken line is the approximation by the function given in this figure. As a result, the hardening modulus H can be expressed as

$$H = h^2\{p_1 \exp(p_2\zeta) + p_3\}, \tag{23}$$

where p_1, p_2, and p_3 are curve-fitting parameters.

Equation (23) is appropriate so long as an initial damage surface of a lamina does not vary with laminate configurations. If the initial damage surface differs from laminate to laminate, equation (23) must be modified. As shown in equation (10), the initial damage surface primarily depends on m_1 and m_2. By changing the value of m_1 and m_2, various shapes of the initial damage surface could be produced. However, we assume that the initial damage surfaces for different laminate configurations have similar shape. In such a case, the difference of laminate configuration is reflected by the size of the surface. When the initial damage surface is enlarged, the released energy in a cracking layer due to new crack formation will be greater than the original value. For the same damage state, the released energy density for these different initial damage surfaces has the following relation [11].

$$\overline{\zeta} = R^2 \zeta, \tag{24}$$

where $\overline{\zeta}$ is the cumulative released energy density concerned with the enlarged initial damage surface, and R is the axial magnification ratio of the initial damage surface, which could be determined experimentally. As has been mentioned previously, $d\sigma_T'/d\varepsilon'^i$ is uniquely related with the damage level, then $d\sigma_T'/d\varepsilon'^i$ for the enlarged initial damage surface should be expressed as follows.

$$\frac{d\sigma_T'}{d\varepsilon'} = p_1 \exp\left(\frac{p_2\overline{\zeta}}{R^2}\right) + p_3. \tag{25}$$

Finally, the complete form of the hardening modulus is given by

$$H = h^2\left\{p_1 \exp\left(\frac{p_2\overline{\zeta}}{R^2}\right) + p_3\right\}. \tag{26}$$

Incorporating equation (26) and equation (13), the constitutive behavior of a cracking layer is completely described.

4. APPLICATIONS OF PRESENT MODEL

The constitutive equations for a cracking layer derived in the previous chapter is applied to the classical lamination theory to predict the constitutive behavior of laminates with a cracking layer. Until the appearance of matrix cracks, the conventional lamination theory is used to compute the stress state of each layer. Once the stress state of a certain layer reaches its inherent initial damage surface, a crack will arise in that layer, and the cracked layer is replaced by an equivalent uniform work-softening layer. Generally, it is impossible to combine elastic constitutive equations with plastic constitutive ones. However, within a small load increment, a plastic material is thought to be a linear elastic material which has a present slope; thus it is reasonable to treat a cracking layer as a linear elastic material during a small load increment. According to the above considerations, the derived constitutive equations for a cracking layer are applied to the lamination theory step by step [11].

4.1. Stress–strain curves

Uniaxial tension tests on GFRP laminates $[0/\theta_4/0]$ (θ = 90, 75, 60, 45) were performed to demonstrate the applicability of the present model. The materials are the same as that used in Fig. 2. The specimens were cut off from GFRP laminates fabricated by an autoclave system. The specimen dimensions are given in Fig. 8. Each specimen was tested in tension parallelly to the fiber direction of the outer plies at a crosshead speed of 0.2 mm/min. The average strain of the middle part of the specimen (100 mm) was measured by the attached extensometer to evaluate the average stress–strain curve of that part. The measured stress–strain curves are shown in Fig. 9 with the predictions by the present model. The parameters used for these predictions are also shown in this figure.

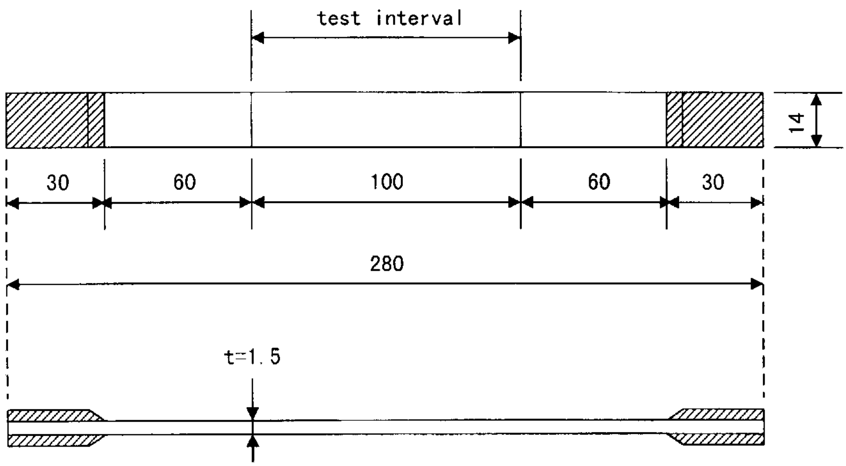


Figure 8. Specimen geometry of GFRP laminate $[0/\theta_4/0]$. The aluminum plates were bonded to the ends of each specimen for reinforcement.

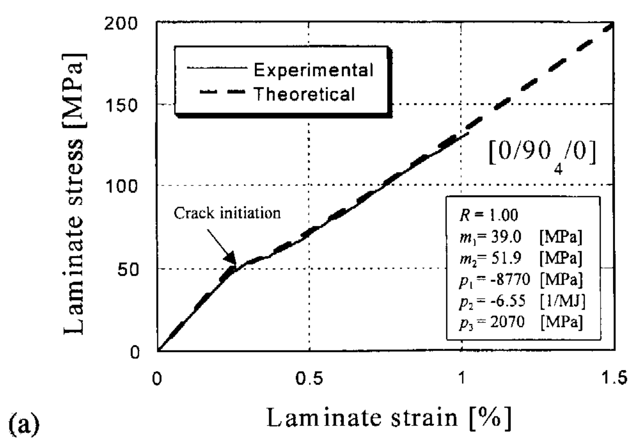


Figure 9. Comparisons of the stress–strain curves of GFRP laminates between experiments and the predictions.

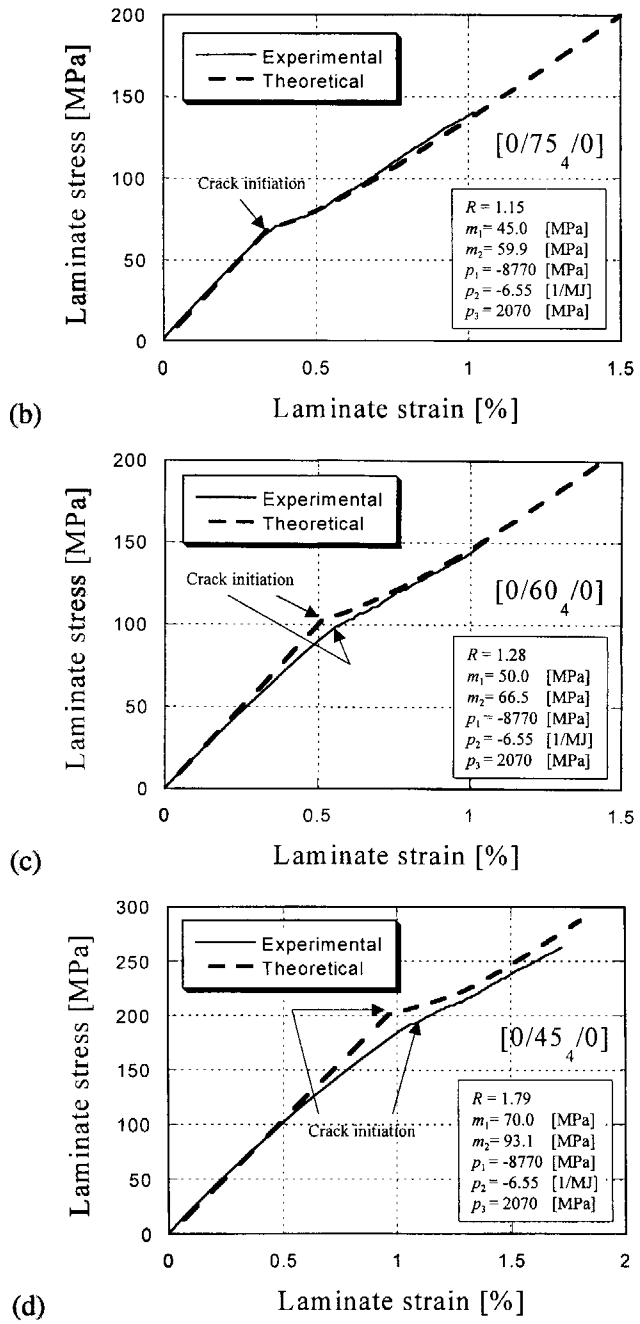


Figure 9. (Continued).

For each laminate configuration, a proper size of the initial damage surface, which has the same aspect ratio as the lamina fracture surface shown in Fig. 3, is selected in such a way that the stress state of the center ply at first crack formation lies on the initial damage surface. The axial magnification ratio R is determined with the selected initial damage surfaces. It should be emphasized that no adjustable parameters are employed to obtain these theoretical predictions. The predictions are in good agreement with experiments even though relatively poor agreements are shown for the laminates with the inclined center ply angle. This may be because of the nonlinear elastic properties of laminae in shear deformation [12].

4.2. Crack density

As has been mentioned previously, if the strain energy release rate in a cracking layer is adopted as the criterion for crack formation, the cumulative released strain energy in a cracking layer would be a proper variable reflecting the present damage level of that layer. Thus, the cumulative released strain energy has the same physical meaning as the number of cracks. Since the cumulative released strain energy in a cracking layer can be continuously calculated, it is easy to predict the variation of the number of cracks during the loading history. The released strain energy by formation of a matrix crack (U_{R1}) is obtained by multiplying the critical energy release rate (G_{mc}) and the newly generated crack area (ΔA).

$$U_{R1} = G_{mc}\Delta A. \tag{27}$$

The number of cracks at the present state is calculated by dividing the cumulative released strain energy by U_{R1}. Here, it should be noted that the critical values of the strain energy release rate differ from laminate to laminate. As has been shown previously, the released strain energy of a lamina with the enlarged initial damage surface due to one crack formation is R^2 times greater than the original value (R is the axial magnification ratio of the initial damage surface). Thus, different critical values of the strain energy release rate should be used for different laminate configurations.

The same type of GFRP laminates as that used in the previous section were tested in tension at a crosshead speed of 0.2 mm/min to investigate the validity of the predicted results for crack density. The specimen dimensions are the same as that shown in Fig. 8. The middle part of the specimen was selected as the test interval, and a crack formation in this area was detected with the naked eye. When a crack formed in the test interval, the corresponding applied load was recorded to evaluate the dependence of the crack density on the applied stress. It should be stated that, in experiments, most matrix crack span the entire cross-section of the center ply, and significant delamination was not detected until saturation of cracks was achieved (see Fig. 10). Figure 11 shows the predicted results of crack density with the experimental results. The critical values of the strain energy release rate are 180, 240, 296, and 580 J/m^2 for [0/90$_4$/0], [0/75$_4$/0], [0/60$_4$/0], and [0/45$_4$/0]. It is found that in all laminates, the predictions are in good agreement with experimental results except for [0/45$_4$/0]. The discrepancy observed in [0/45$_4$/0] may arise from the significant nonlinear elastic behavior of laminae in shear [12]. From these close agreements between the predictions and experimental data, it is concluded that a cracking layer has the same properties as a work-softening material in a quantitative sense.

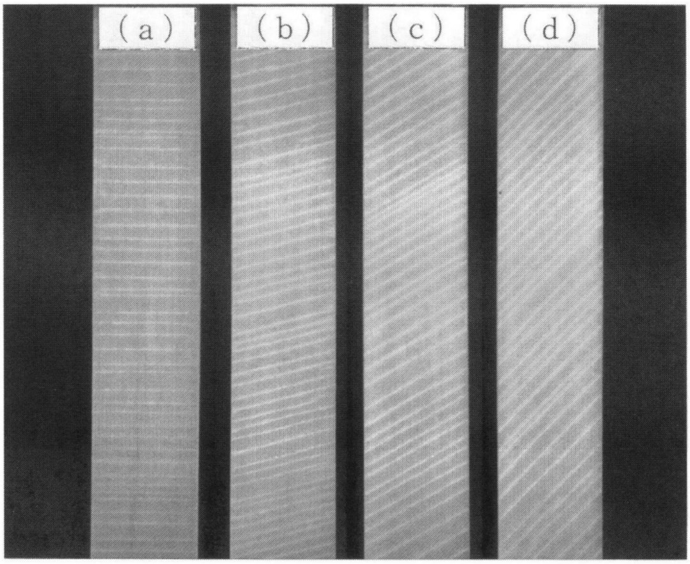


Figure 10. Appearance of cracked specimens. (a) [0/90$_4$/0]; (b) [0/75$_4$/0]; (c) [0/60$_4$/0]; (d) [0/45$_4$/0].

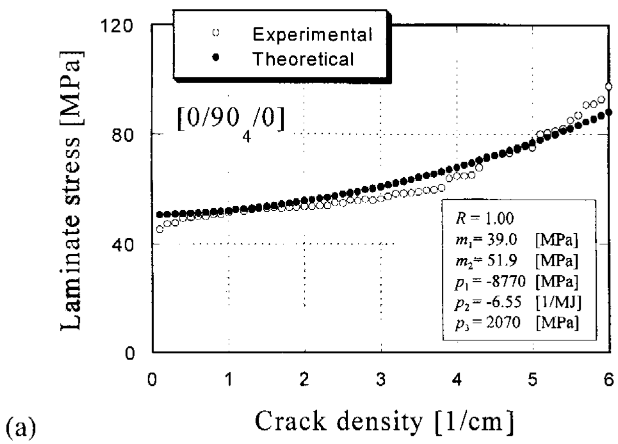
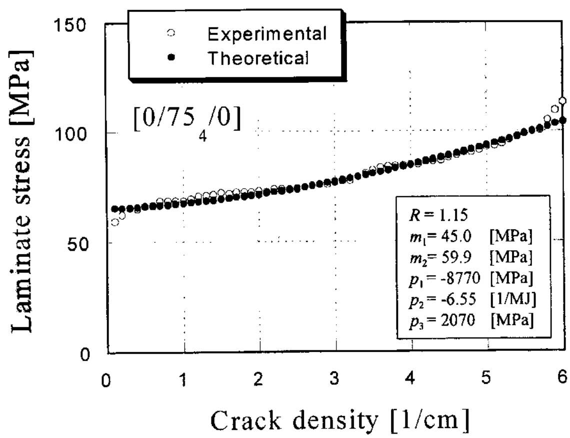
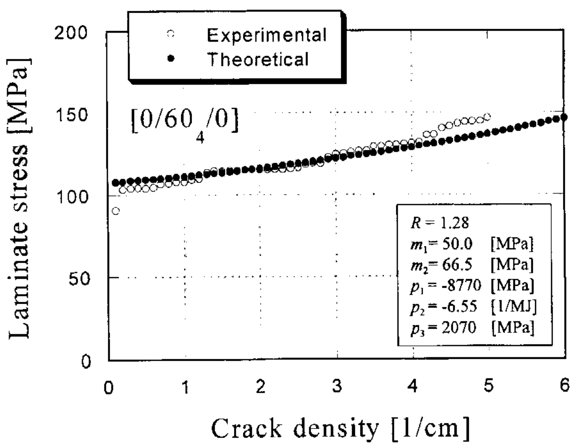


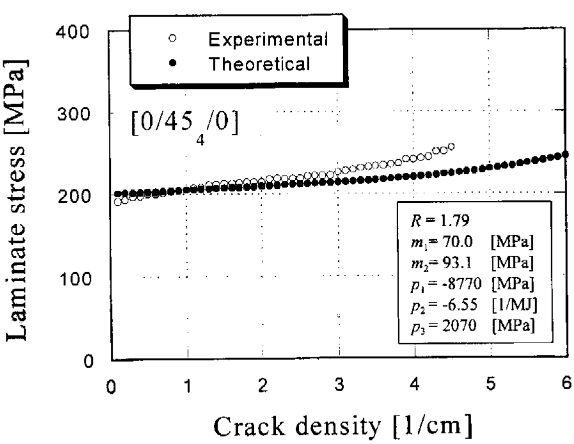
Figure 11. Comparisons of crack density of GFRP laminates between experiments and the predictions.



(b)



(c)



(d)

Figure 11. (Continued).

5. CONCLUSION

In this work, the constitutive model for predicting the nonlinear stress–strain relationship of general-angleply laminate with cracking layers was developed. First, the average behavior of a cracking layer was investigated with the experimental results on crossply laminates. The observed constitutive behavior of the 90° ply in a crossply laminate was found similar to the expected results for a work-softening material. Under this experimental observation, we proposed the new concept that a cracking layer behaves as a uniform work-softening layer in general laminate configurations. The predictions of the crack density as well as the stress–strain curves were made with this new concept, and excellent agreements were obtained between experiments and the predictions for all laminate configurations, despite the lack of any adjustable parameters, which are commonly used to fit analytical predictions to a particular set of experiments. Even though the present model requires further investigation, especially for more appropriate definition of a damage surface, the present model provides a useful scheme to formulate the macroscopic nonlinear behavior of a laminate with cracking layers.

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